# The Three-Body Problem and Space Missions

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**The three-body problem is of utmost importance in the field of celestial mechanics and the planning of space missions. The study of how three bodies interact with one another when subjected to mutual gravitational pull is referred to as the "three-body problem." A general three-body problem needs to be studied for the study of celestial mechanics. However, a simplified form of the problem known as the restricted three-body problem is enough for designing space missions because the influence of the spacecraft or satellite on the motion of celestial bodies is small. In this paper, the issue associated with the three-body problem has been discussed. The analytical and numerical methods of tackling the problem have both been covered in this discussion. In addition, the use of the restricted three-body problem in the planning of space missions has been considered.**

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# I. Introduction

In the three-body problem, three bodies move through space according to their mutual gravitational interactions, as outlined by Newton's theory of gravity. For this problem to be solved, future and past motions of the bodies must be uniquely calculated based merely on their current positions and velocities. In general, the motions of bodies occur in three dimensions (3D), and neither their masses nor their initial conditions are constrained. Consequently, this is known as the *general three-body problem*. The problem's intricacy is not readily apparent, especially when one considers that the two-body problem has well-known closed-form solutions expressed in elementary functions. Adding one more body to the problem makes it impossible to achieve similar answers. The equations are non-integrable, and the solution becomes susceptible to the initial conditions causing it very difficult to use numerical methods to understand the evolution of the state of the system. In celestial mechanics, the general three-body problem attempts to predict the motions of gravitationally interacting astronomical bodies.

The three-body problem can be significantly simplified for the motion of spacecraft/satellites in space because of the negligible mass of the satellite compared to the planets and stars. This is because the gravitational influence of the satellite on the planets and stars is negligible. If this condition is satisfied, the general three-body problem becomes the *restricted three-body problem*. In this case, the two dominant masses move in either circular or elliptical orbit around their COM.

The three-body problem has been studied for over three hundred years. It was formulated and studied by Newton in his Principia. He found the problem difficult to solve; however, he was able to give an approximate solution to the motion of the Moon and the Earth around the Sun. Later, Euler and Lagrange obtained special cases of three body problems in which the three bodies were in a straight line or formed a triangle of equal sides, respectively, and they preserved the configuration. Poincare studied and advanced the solution to the three-body problem in the second half of the nineteenth century. His work demonstrated the non-integrability of the system of equations describing the three-body problem. Poincaré's new methods allowed him to identify the problem's unpredictability and discover the first manifestation of a new phenomenon, which is now commonly known as chaos. Later, extensive studies of the stability of orbits in a special restricted three-body problem proposed by Poincare were done by Levi-Civita, Lyapunov and Birkhoff.

Researchers, realising that closed-form solutions to the three-body problem were unlikely to be discovered, examined infinite series solutions. Lindstedt and Gylden, Painleve attempted to find solutions given in terms of infinite series but failed. In 1912, Sundman found a complete solution to the three-body problem in terms of power series expansion; however, the solution converges very slowly and cannot be used for any practical applications. Although the system is non-integrable, as proved by Poincare, periodic solutions (orbits) can be obtained depending on a set of initial conditions. These orbits were discovered by using analytical methods. Henon and Szebehely were able to find many periodic orbits using computer simulations. Different interesting problems were solved using the simulation. Modern applications of the three-body problem include the trajectory design for satellites/spacecraft while minimising the energy used.

The rest of the paper is organised as follows. Section II introduces and describes the general three-body problem. Section III presents the numerical and analytical attempts to solve the general three-body problem. The restricted three-body problem is presented in Section IV, followed by the existence of libration points in Section V. Section VI provides an overview of applications of the three-body problem in the mission design of spacecraft. The paper is concluded in Section VII.

## II. General Three Body Problem

In general three body problem, three arbitrary masses interact under the influence of gravitational force. The motion of the objects are in 3 dimensions. However, if the motion is restricted to a plane, it is called planar general three body problem. Let $M_i$ be the mass of $i_{th}$ body, where $i = 1, 2, 3$. Their positions with respect to the origin of an inertial Cartesian coordinate system be represented as $\vec{R}_i$ and we define the position of one body with respect to another by $\vec{r}_{ij} = \vec{R}_j - \vec{R}_i$, where $i, j = 1, 2, 3$ and $i \neq j$. Then using Newton's gravitation formulation, the resulting equations of motion are

$$M_i \frac{d^2\vec{R}_i}{dt^2} = G \sum_{j=1}^{3} \frac{M_i M_j}{r_{ij}^3} \vec{r}_{ij} \tag{1}$$

where $G$ is the universal gravitational constant. Eq. (1) represents three coupled, second-order, ordinary differential equation. In scalar form, these can be written as 9 coupled, second-order, ordinary differential equation. However, using state space form, the equation can be transformed into 18 coupled, first-order, ordinary differential equation.

Because of the symmetry of $r_{ij} = -r_{ji}$, summing up the equations of Eq. (1) results in

$$\sum_{i=1}^{3} M_i M_i \frac{d^2\vec{R}_i}{dt^2} = 0 \tag{2}$$

Integrating Eq, (2) twice results in

$$\sum_{i=1}^{3} M_i \vec{R}_i = \vec{C}_1 t + \vec{C}_2 \tag{3}$$

Eq: (3) represents the motion of the COM of the system which is motion in a straight line. Using the conservation of angular momentum around the center gives

$$\sum_{i=1}^{3} M_i \vec{R}_i \frac{d\vec{R}_i}{dt} = \vec{C}_3 \tag{4}$$

And the conservation of energy gives

$$E_t = E_k + E_p = C_4 \tag{5}$$

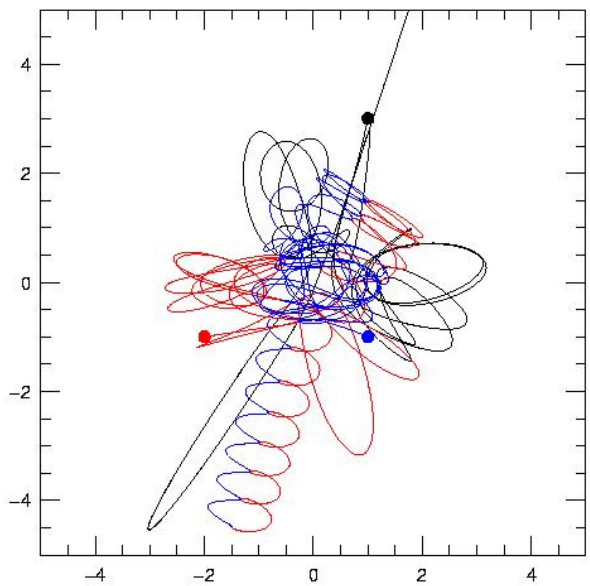


**Fig. 1 Pythagorean three body problem solution**

where $E_t$ is the total energy, $E_k$ is the kinetic energy given by

$$E_k = \frac{1}{2}\sum_{i=1}^{3} M_i \frac{d\vec{R}_i}{dt}.\frac{d\vec{R}_i}{dt} \tag{6}$$

and $E_p$ is the potential energy given by

$$E_p = -\frac{G}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{M_i M_j}{r_{ij}} \tag{7}$$

where $i \neq j$. The constants, $\vec{C}_1, \vec{C}_2, \vec{C}_3$ and $C_4$ represents the 10 integrals of motion in the general three body problem. Since $E_k > 0$ and $E_p < 0$, $E_t = C_4$ can be positive, negative or zero. If $C_4 > 0$, the three-body system must split, which causes one body to get ejected while the remaining two bodies form a binary system. $C_4 = 0$ is a special case in which one body escapes the system. In case of $C_4 < 0$, either one body escapes or periodic orbits are formed based on the moment of inertia of the system.

## III. Methods of Solving General Three Body Problem

In this various methods employed for solving general three body problem has been discussed. Earlier, analytical solutions where obtained with specific conditions. However, with advent of fast computing technology, numerical

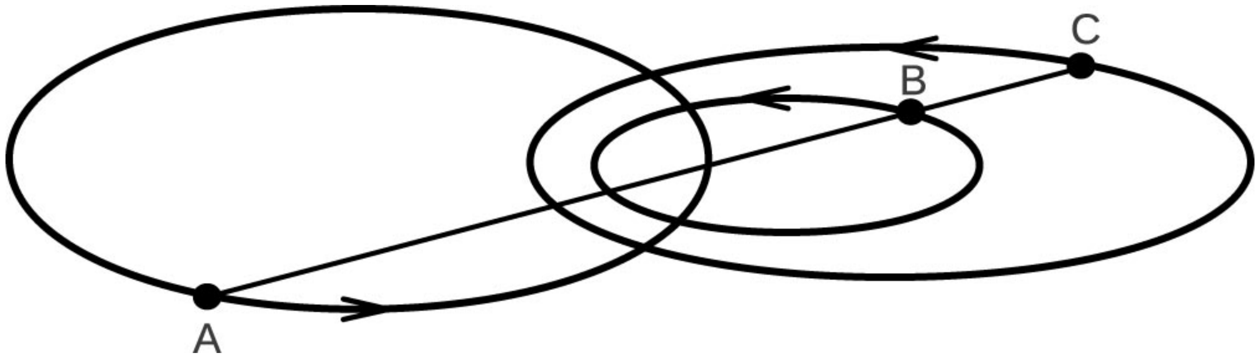



**Fig. 2 Euler solution**

solution has become common with search for a general analytical solution.

### A. Analytical Method

Euler and Lagrange obtained two different classes of periodic solutions to the general three-body problem. Euler [1] examined a linear arrangement of the three bodies and established that they move in elliptical orbits. There was no assumption made about the masses. The initial configuration of the three bodies is maintained if the ratio AB/BC in fig. 2 has a particular value that depends on the masses. Euler demonstrated that the line AC would rotate about the bodies' centres of mass, resulting in periodic elliptical orbits. In addition, he established that the ratio AB/BC would remain unchanged along AC as the bodies moved. Because the three bodies can be arranged in three distinct ways along the line, three solutions correspond to the three possible body bodies. Notably, the Euler solution is unstable against minor displacements.

Lagrange [2] considered a triangle configuration of the three bodies. The initial configuration in the Lagrange solution is an equilateral triangle with three bodies positioned at its vertices. Lagrange demonstrated that, under ideal initial conditions, the initial configuration is retained (becomes the central configuration) and the orbits of the three bodies stay elliptical throughout the motion (see fig. 3). The triangle remains equilateral, although the core configuration is maintained, and its size and direction change as the bodies travel. Since the triangle can be orientated in two distinct ways, two solutions correspond to its orientation.

These unique configurations (known as the central configurations in celestial mechanics) plays a crucial role in orbital stability research. Several other periodic solutions has been found. Hill [3] also found periodic orbits known as Hill problem. Major contribution was given by Poincare [4]. He developed a method to find periodic orbits in circular restricted three body problem which can be applied to the general three-body problem. According to the theorem developed by him and contemporary scientists, it was shown that the periodic solutions existed even for non-integrable Hamiltonian systems.

Periodic orbits in 3 dimensional general three body problem are typically determined numerically [5–7]. The general

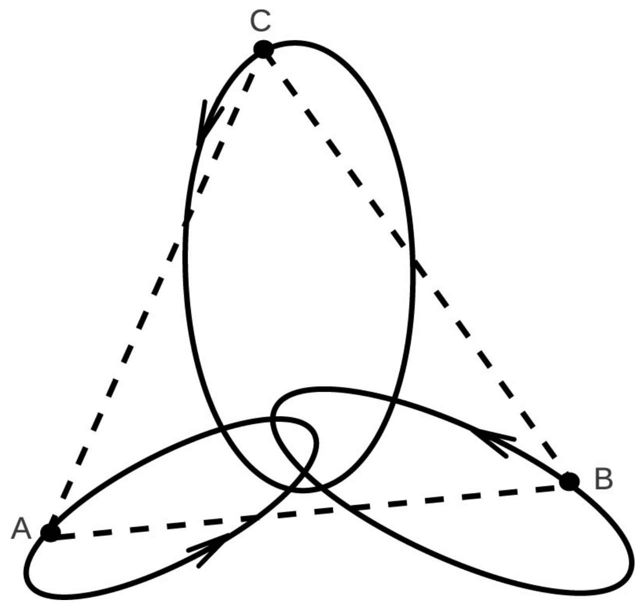



**Fig. 3 Lagrange solution**

three-body problem is formulated suitably for efficient numerical computations. Recently, analytical and numerical methods have been used to get new periodic solutions. Moore [8], Chenciner and Montgomery independently discovered the 8-type periodic solution with equal masses, fig. 4. Furthermore, the KAM theorem was used to establish the stability of the solution [9].

### B. Numerical Methods

Symplectic integration is the most often utilized numerical approach. This method depends on the mathematical parameterization of the equations of motion into matrices and assumes a low occurrence of events in which the orbital velocity could rapidly increase. This is an implicit assumption in celestial mechanics regarding the potential values of eccentricity for the system under consideration. With low eccentricity, greater time steps can be employed, which expedites calculations greatly. Due to the symplectic character of Hamiltonian dynamics, in which matrices are symmetric and easily invertible, this is conceivable. This mathematical characteristic can investigate nonlinear Hamiltonian dynamics using computationally efficient mapping integration methods.

One might imagine a fundamental mapping technique, such as Euler's or leapfrog's integration, applied with symplectic features to provide greater speed and precision. Other extensively used computing approaches are Runge–Kutta similar techniques. Different approaches to the general algorithm of Runge–Kutta have obtained the coefficients for the Runge–Kutta method to high order, albeit at the expense of computing speed. Taylor's method of integration, which relies on the knowledge of terms within a Taylor series, is a more traditional approach to integration.

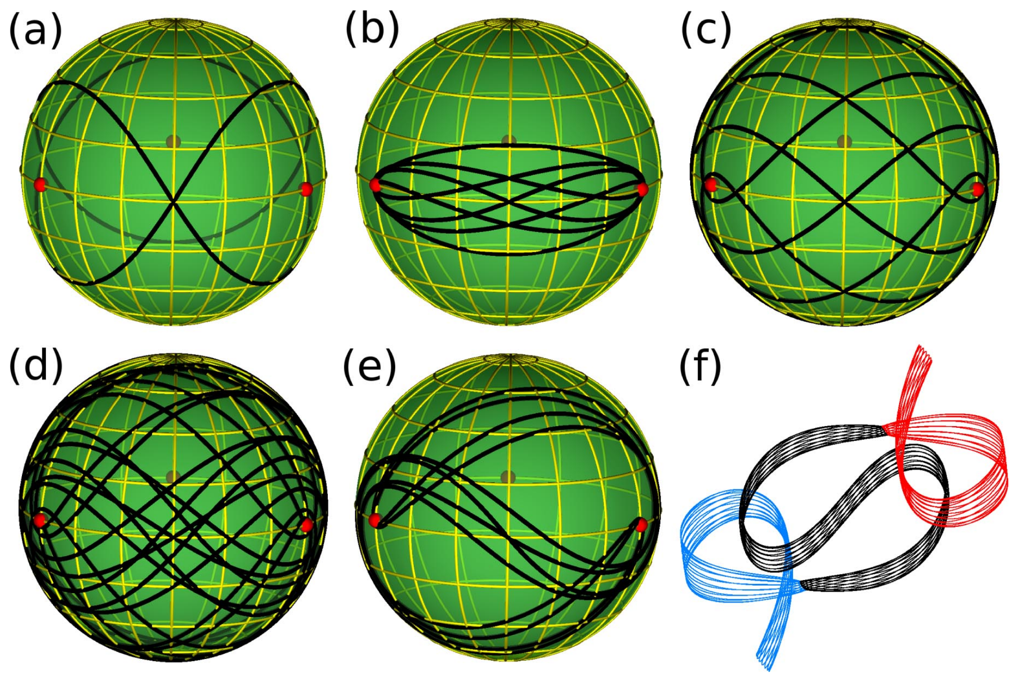



**Fig. 4 New Periodic Solutions. (a) The figure-8 orbit; (b) butterfly orbit; (c) moth orbit; (d) yarn orbit; (e) yin-yang orbit; (f) the yin-yang orbit in real space[10]**

## IV. Restricted Three Body Problem

This section discusses the circular restricted three-body problem (CR3BP), which is a common occurrence in our universe. In CR3BP, the mass of one object is significantly lesser than the masses of the other two objects, i.e., $M_1 >> M_3$ and $M_2 >> M_3$. This results in the motions of $M_1$ and $M_2$ being limited to circular orbits around their centre of mass. In literature, masses $M_1$ and $M_2$ are referred to as primaries. Since the motion of $M_1$ and $M_2$ follows a two-body problem, the solution for their motion is known, which can be used to determine the motion of the third body. Hence, the problem becomes simpler.

Let $\zeta_1(x, y, z)$ and $\zeta_2(x^*, y^*, z^*)$ represents the inertial and rotating coordinate systems respectively as described in fig. 5. $\zeta_2$ is associated with the frame of rotation of the $M_1$ and $M_2$ masses. The centre of mass of $M_1$ and $M_2$ is chosen as the origin of both frames.

Let $\omega$ be the angular frequency of the orbit of $M_1$ and $M_2$. Defining $\mu = \frac{R_1}{R_1+R_2}$, $\alpha = \frac{R_2}{R_1+R_2}$, $\vec{r}_i = \vec{R}_i - \vec{R}_3$. Then, using Newton's Gravitational formulation, the equations of motion of the third body is written in $\zeta_1$ as:

$$\ddot{x} = -\frac{\alpha}{r_1^3}(x - \mu\cos(\omega t)) - \frac{\mu}{r_2^3}(x + \alpha\cos(\omega t)) \tag{8}$$

$$\ddot{y} = -\frac{\alpha}{r_1^3}(y - \mu\sin(\omega t)) - \frac{\mu}{r_2^3}(y + \alpha\sin(\omega t)) \tag{9}$$

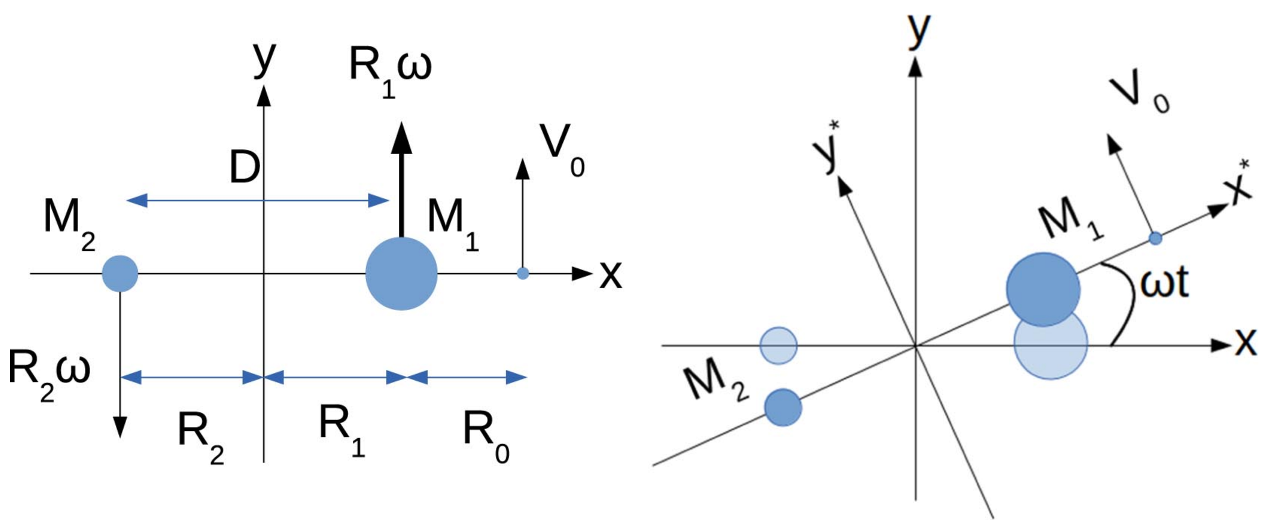



**Fig. 5 Inertial and Rotating Frames[11]**

$$\ddot{z} = -\left(\frac{\alpha}{r_1^3} + \frac{\mu}{r_2^3}\right) \tag{10}$$

These set of equations can also be written in the rotating coordinate system ($\zeta_2$) as [11]:

$$\ddot{x}^* - 2\dot{y}^* = x^* - \frac{\alpha}{r_1^3}(x^* - \mu) - \frac{\mu}{r_2^3}(x + \alpha) \tag{11}$$

$$\ddot{y}^* + 2\dot{x}^* = \left(1 - \frac{\alpha}{r_1^3} - \frac{\mu}{r_2^3}\right) y^* \tag{12}$$

$$\ddot{z}^* = -\left(\frac{\alpha}{r_1^3} + \frac{\mu}{r_2^3}\right) z^* \tag{13}$$

where $r_1 = D\sqrt{(x^* - \mu)^2 + y^{*2} + +z^{*2}}$ and $r_2 = D\sqrt{(x^* + \alpha)^2 + y^{*2} + +z^{*2}}$.

## V. Libration Points

Lagrange used the Eqs. (11)-(13) to describe the equilibrium positions of the third body. These points are called Lagrange or Libration points. In equilibrium,

$$x^* - \frac{\alpha}{r_1^3}(x^* - \mu) - \frac{\mu}{r_2^3}(x + \alpha) = 0 \tag{14}$$

$$\left(1 - \frac{\alpha}{r_1^3} - \frac{\mu}{r_2^3}\right) y^* = 0 \tag{15}$$

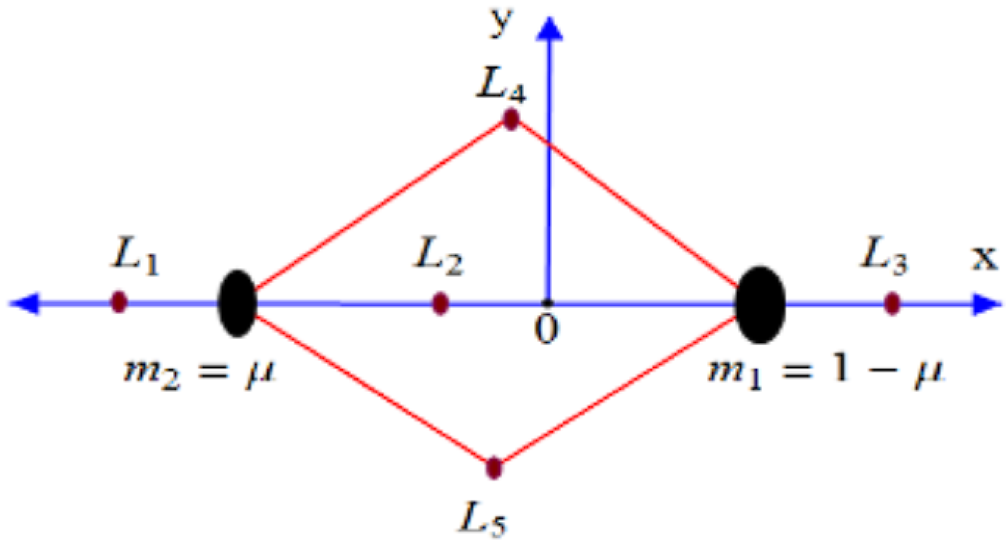



**Fig. 6 Equilibrium Points (Lagrange Points)**

$$\left(\frac{\alpha}{r_1^3} + \frac{\mu}{r_2^3}\right) z^* = 0 \tag{16}$$

In eq. (16), if $z^* \neq 0$, then $y^* = 0$. However, if $z^* = 0$, $y^* \neq 0$. Hence, the equilibrium solutions are confined to a plane. Now, since $y^* \neq 0$, $\left(1 - \frac{\alpha}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0$ resulting in $r_1 = r_2 = 1$. These form an equilateral triangle and the two equilibrium points are called the Lagrange triangular $L_4$ and $L_5$ points as shown in fig. 6. These equilibrium points are stable for mass ratios $0 \leq \mu \leq \mu_0$ where $\mu_0 = 0.0385208965$ is called Routh's critical mass. Eq. (14) gives 3 co-linear equilibrium points located on a line passing through the primaries. These solutions are named $L_1$, $L_2$ and $L_3$ as shown in fig. fig:7 and are unstable.

Using the potential function $\phi^*$ defined as

$$\phi^* = \frac{1}{2}[x^{*2} + y^{*2}] + \frac{\alpha}{r_1} + \frac{\mu}{r_2} \tag{17}$$

we define Jacobi Constant, $C^*$ in rotating coordinate system as

$$C^* = \mu + 2\mu\rho_0 + \frac{1-\mu}{\rho_0} + \frac{\mu}{1+\rho_0} + 2\sqrt{\rho_0(1-\mu)} \tag{18}$$

where $\rho_0 = R_0/D$ represents the normalized initial position of the third mass. Note that with increase in $\rho_0$ (fig. 7), the forbidden zones and eventually shrink to the two Lagrange points, $L_4$ and $L_5$. Lagrange points are of immense significance for placing telescopes and satellites that need to be present at a fixed position.

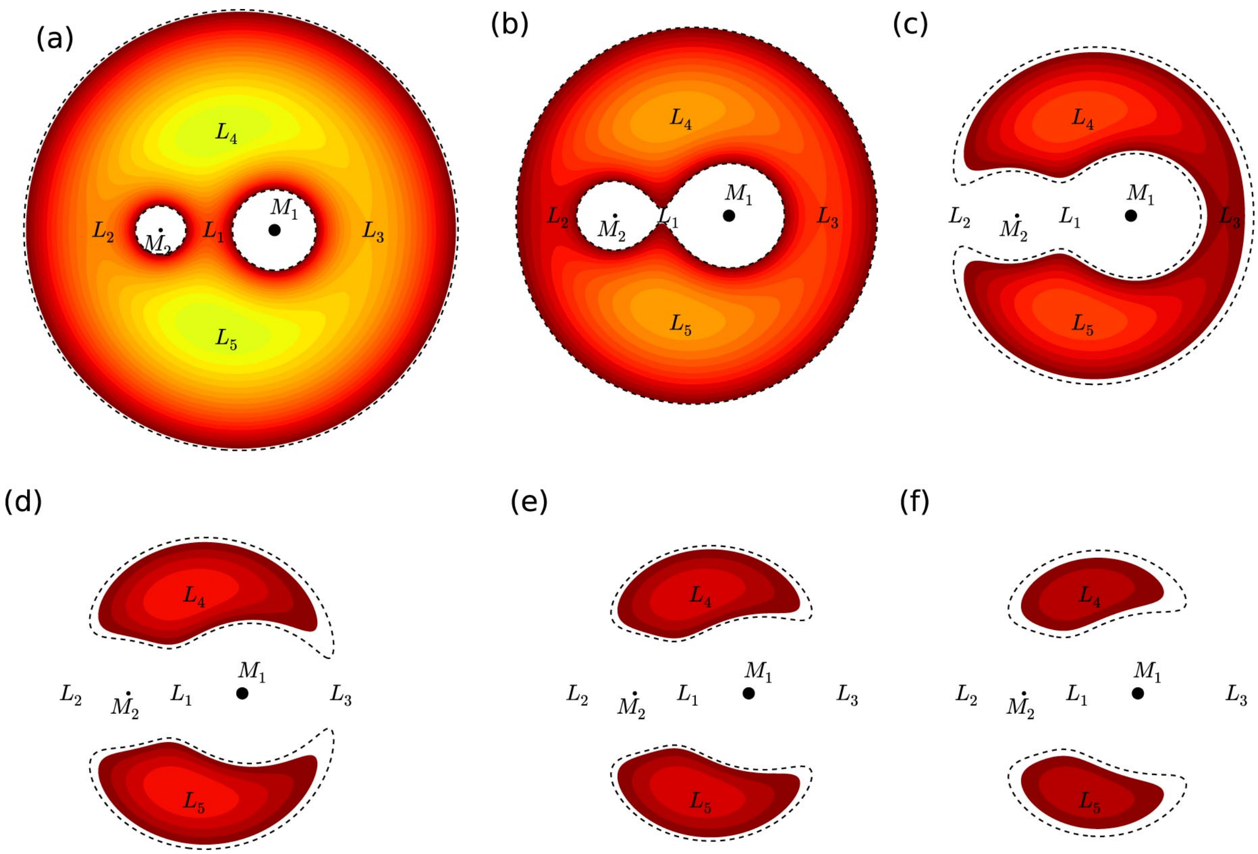
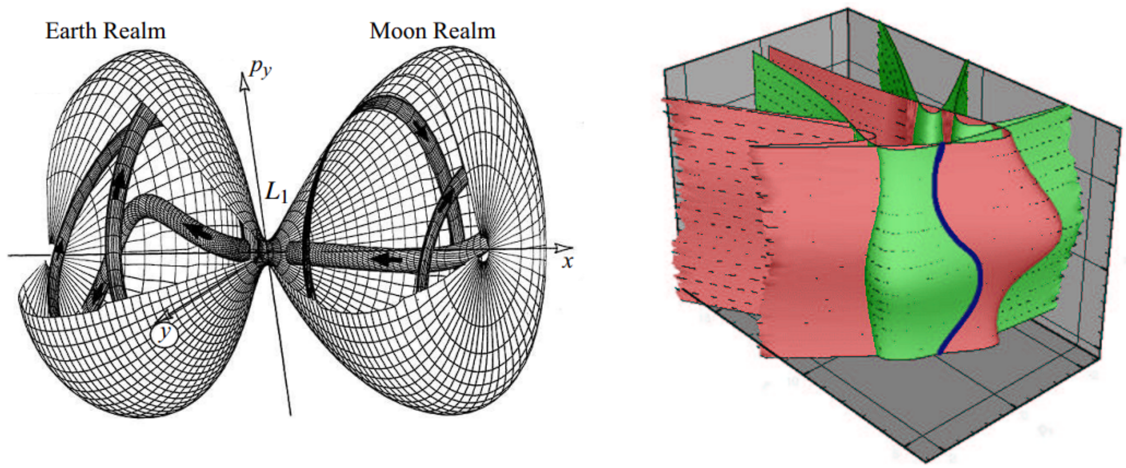



**Fig. 7 Forbidden Zones in the rotating coordinate system, computed for a mass ratio $\mu$ = 0.3 and various initial positions, $\rho_0$ of the third mass. The dashed lines represent the boundary of zero velocity while the colored region indicates the forbidden zones. (a) $\rho_0$ = 0.2; (b) $\rho_0$ = 0.28; (c) $\rho_0$ = 0.36; (d) $\rho_0$ = 0.44; (e) $\rho_0$ = 0.52; (f) $\rho_0$ = 0.6. [10]**




(a) (b)

**Fig. 8 Tube dynamics and lobes dynamics describe the global transport in phase space in a multi-scale way. a) Tubes connecting realms. b) Lobe structure. [12]**

## VI. Space Mission Design

Restricted three-body problem is the simplest and most elegant form of three-body problem which comes with many advantages for practical applications. In this section, some applications in space mission design has been discussed.

### A. Tube and Lobe Dynamics

Tube and Lobe Dynamics is an attempt to design low energy pathways in our solar system. This allows for orbits around several planetary objects instead of just flybys. The Hamiltonian for many systems takes the form of the kinetic plus effective potential. The potential's shape reveals information about unstable and stable equilibria, as well as the geometry of realms surrounding equilibria, which split the energy manifold at the coarsest level. The energy value determines the connectedness of the realms. Transport across realms in a rank one saddle is mediated by phase space tubes restricted by stable and unstable manifolds connected with typically hyperbolic invariant manifolds surrounding unstable equilibria. The tube dynamics theory presented in multiple works can be utilised as the foundation for a statistical theory for computing transit rates between realms.

Lobe dynamics are also required to investigate transfer between subsets within a specific domain. It provides a geometric framework for debating, characterising, and quantifying ordered structures crucial to realm transfer. Tube dynamics, in conjunction with lobe dynamics, have provided a complete view of global phase space transport for systems with two degrees of freedom. This accomplishment motivates researchers to expand the combination of approaches to 2 DOF, which preliminary studies indicate is feasible.

These new works represent the first step toward a precise global transport theory. An image like this involves the

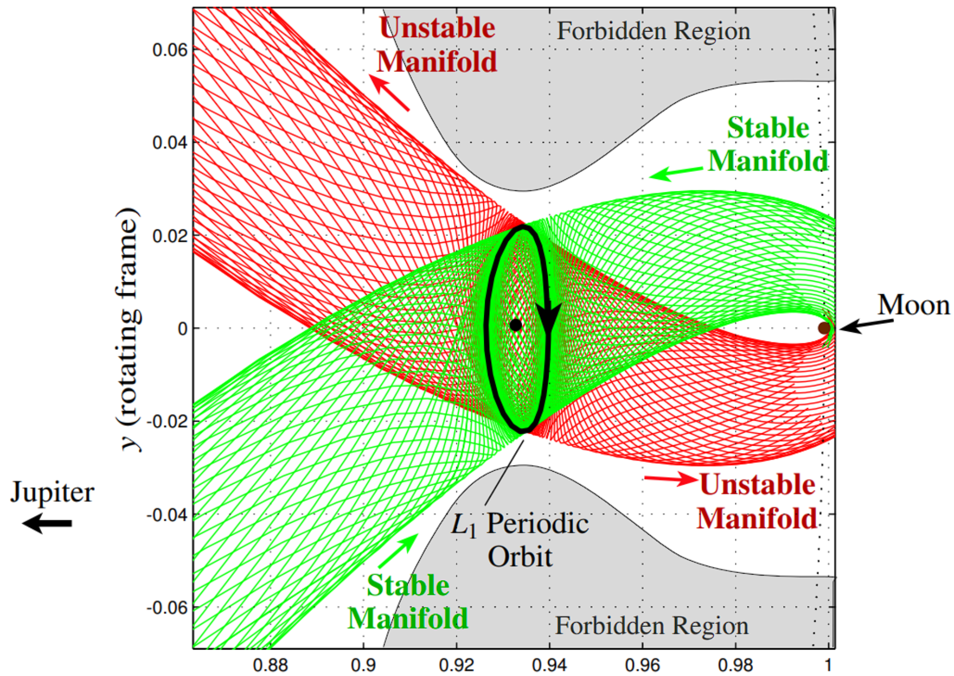



**Fig. 9 Stable and Unstable Manifolds [13]**

integration of tube dynamics and lobe dynamics. Koon's key theorem argues that tube dynamics and lobe dynamics are inextricably related, and numerical investigations confirm this. Such unification is desperately required. Chemists have studied molecular dynamics using tube and lobe dynamics variations, but these two separate views should merge.

The combination of virtually invariant set approaches with invariant manifold, and lobe dynamics techniques for systems with two degrees of freedom has made promising progress. For example, dynamical systems techniques have been utilised to identify where box refinements are required, significantly speeding up the computation.

In the three-body problem of orbital mechanics, a halo orbit is a periodic, three-dimensional orbit close to one of the $L_1$, $L_2$, or $L_3$ Lagrange points. At each Lagrange point, there are continuous "families" of both northern and southern halo orbits. Halo orbits are prone to instability, therefore station keeping may be required to maintain a satellite in orbit.

### B. James Webb Space Telescope (JWST)

The James Webb Space Telescope, often known as JWST, is currently located in a halo orbit around the Sun-Earth L2 Lagrange point. This point is approximately 1,500,000 kilometres (933,000 miles) beyond the circumference of the Earth's orbit around the Sun. It is able to accomplish this by orbiting at a distance from L2 that is between 250,000 and 832,000 kilometres (155,000 and 517,000 miles), which keeps it clear of the shadows cast by both the Earth and the Moon. While the Moon is around 400,000 kilometres (about 250,000 miles) from Earth, Hubble is orbiting approximately 550 kilometres (about 340 miles) above the surface of the planet. In comparison, the distance between the Moon and Earth is approximately 250,000 miles. The telescope is able to maintain a nearly constant distance while

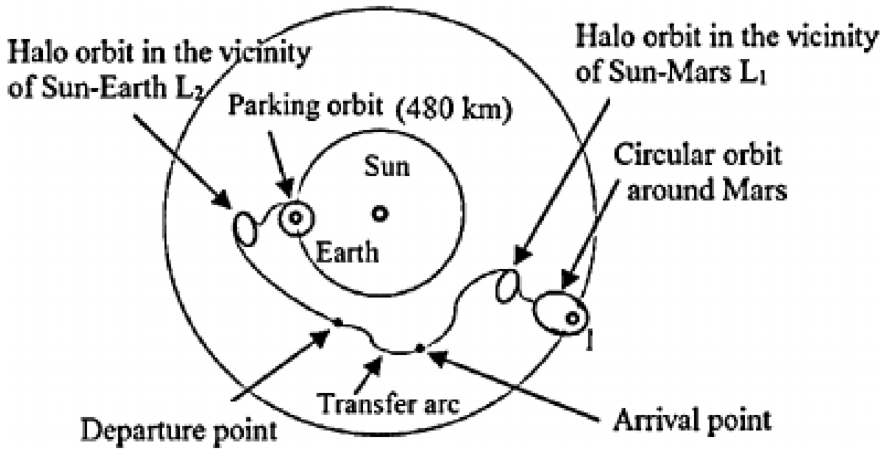



**Fig. 10 Mars interplanetary trajectory design via Lagrangian points [14]**

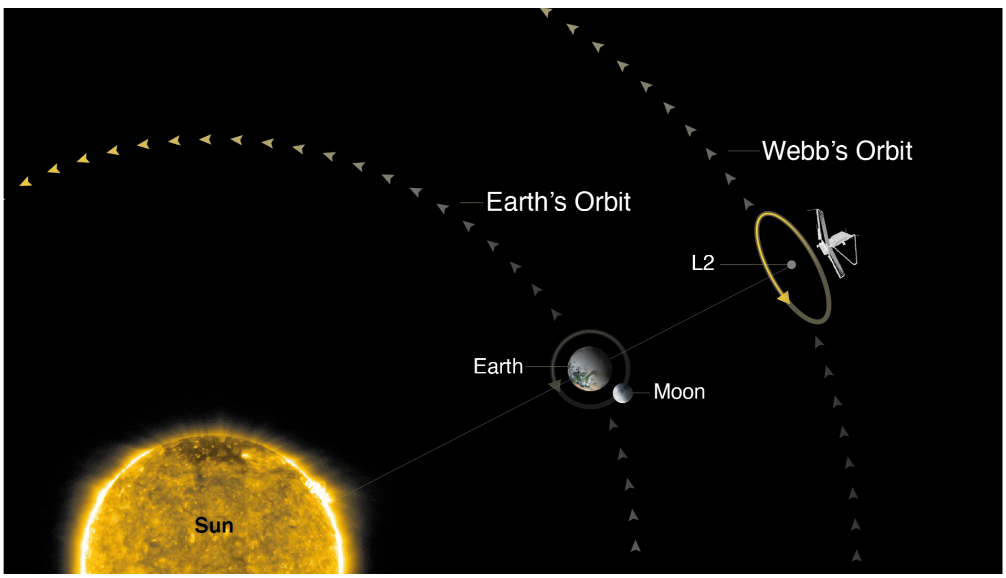



**Fig. 11 Halo Orbit of JWST at $L_2$ of Sun-Earth system [NASA]**

continuously orienting its distinctive sun shield and equipment toward the Sun, Earth, and Moon because there are objects that can orbit the Sun and the Earth in synchronisation with this Sun-Earth L2 point. This is made possible because of objects that can orbit the Sun and the Earth in synchronisation with this Sun-Earth L2 point. This is made possible by the fact that objects can orbit the Sun and the Earth in sync with each other. This allows for a greater degree of precision in astronomical calculations.

The structure of the telescope is designed to block out any heat or light that may come from the Sun, the Earth, or the Moon. In addition to this, it prevents even the slightest temperature variations from occurring as a result of the shadows created by the Earth and Moon, which may have a negative impact on the construction if they occurred. On the side of the telescope that is facing the Sun, all of this is achieved while ensuring that solar power and communications with Earth continue uninterrupted. It is now possible to do this thanks to the spacious orbit that the telescope maintains, which enables it to avoid casting shadows. Because of this design, the temperature of the spacecraft is maintained at or below the threshold of 50 K, which is necessary for observations in the dim infrared. The cutoff temperature is equal to 223 °C (370 °F).

**C. Solar and Heliospheric Observatory (SOHO)**

At this time, the SOHO spacecraft travels through space while circling the Sun-Earth $L_1$ point in a halo orbit. The spacecraft can see the Sun and Earth from various perspectives because of its orbit. This is the point between the Earth and the Sun at which the balance of the Sun's (larger) gravity and the Earth's (smaller) gravity is equal to the centripetal force that is required for an object to have the same orbital period in its orbit around the Sun as the Earth, with the result that the object will stay in that relative position. The Sun's gravity is greater than the Earth's gravity, but the Earth's gravity is smaller than the Sun's gravity. The gravity of the Sun is far higher than that of the Earth, whereas the Earth's gravitational pull is considerably weaker than that of the Sun. The gravitational pull of the Earth, which is analogous to the centrifugal force, helps to ensure that the gravitational pull of the Sun is held in check.

This is accomplished by the satellite making slow rotations around the First Lagrangian Point (L1), which is the location at which the combined gravitational pull of the Earth and Sun maintains SOHO in an orbit that is aligned with the line connecting the Earth and the Sun. This is where the combined gravitational pull of the Earth and Sun maintains SOHO in orbit aligned with the line connecting the Earth and the Sun. This is the point in space where the combined gravitational force of the Earth and Sun keeps SOHO in an orbit that is aligned with the line that runs between the Sun and the Earth. As a direct result, SOHO is in a position to trace the path that Earth travels around the Sun as it completes one revolution after another. The distance between the Earth and the $L_1$ point, which rotates anticlockwise about the Sun, is roughly 1.5 million kilometres. The $L_1$ point is located around 1.5 million kilometres away. This distance is nearly three and a half times as great as the one that separates the Earth and the moon. There, SOHO is provided with a view of our midday star that is free from any and all sorts of noise as well as any other interference that may be there. All of the early solar observatories revolved around the Earth, so their observations had to be constantly disrupted. This was necessary since the Earth 'eclipsed' the Sun regularly, which rendered the observations worthless and required the observatories to do so.

The SOHO spacecraft is not precisely positioned at $L_1$ because doing so would render communication impossible owing to radio interference induced by the Sun and because this orbit would not be stable. Instead, the SOHO spacecraft is in an orbit that is somewhat closer to L1. On occasion, though, it will be described as being at the $L_1$ level. Despite this, the SOHO spacecraft is typically considered to be in the $L_1$ orbital position when speaking of its current location. On the other hand, the orbital location that is frequently referred to as the $L_1$ orbital location is the place where it is believed that the SOHO spacecraft is now located. Instead, it is found in the plane that is constantly moving through L1, and that is perpendicular to the line that joins the Sun and the Earth. This plane is called the $L_1$ plane. The $L_1$ plane

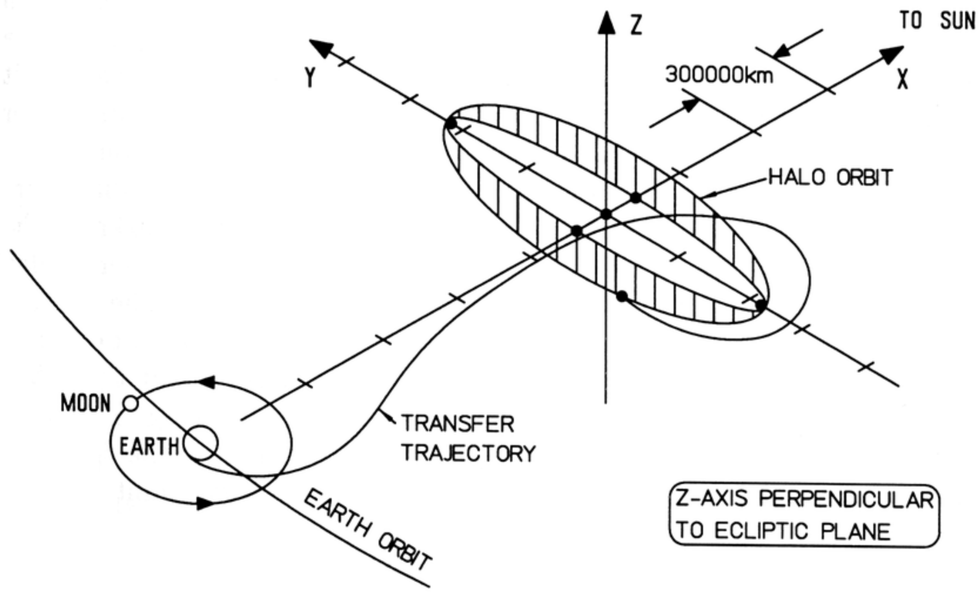



**Fig. 12 Halo Orbit of SOHO Spacecraft at $L_1$ of Sun-Earth system [ESA]**

is the name given to this particular aircraft. The ecliptic plane is the name that is given to this particular plane when discussing its technical characteristics. It will continue moving forward in this plane as it forms an oval halo orbit, with $L_1$ serving as the focal point of the orbit. As a result of the fact that its velocity is coupled to that of Earth, it finishes an orbit around $L_1$ once every six months, whereas $L_1$ finishes an orbit around the Sun once every year. The fact that the velocity of $L_1$ is connected to the velocity of Earth is the root cause of this difference. This is because the motion of $L_1$ is joined with that of the Earth, which causes this phenomenon to occur. This ensures that the SOHO spacecraft will always be in a position from which it can communicate with Earth in a fashion that is both effective and efficient at the same time.

## VII. Conclusion

In this paper, the research carried out in the field of the three-body problem has been reviewed and summarized. Three body problem is a mathematical problem related to Newton's law of gravitation. The general and restricted three-body problems are studied with solutions and their importance. Though it is a classical problem, which has been studied for hundreds of years, the general three-body problem cannot be solved by applying present mathematical methods. Instead, approximations are used to obtain a solution which gives the relative positions of the three bodies at any given time. The restricted three-body problem simplifies the problem and has practical applications. The problem results in libration points which are used for the placement of scientific equipment. Libration points occur where the gravitational influence of two large bodies is approximately equal, causing an object there to remain stationary relative to those bodies. Furthermore, the description of the use of this problem for planning the positioning of JWST and

SOHO has been discussed, followed by the importance of the general three-body problem in the design of tubes and lobes for space travel with minimum energy requirements.

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